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ON PATH INDEPENDENT RANDOMIZED CHOICE

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IN LIGHT OF THE famous impossibility theorem of Arrow involving "rational" social choice functions (i.e. choice functions generated by underlying preference relations over an alternative set X)[2] and Arrow's subsequent comments [**1**, p. 120], Plott [**4**] has suggested the concept of "path independence" of a choice function $c(\cdot)$ (i.e. for all $S, T \subseteq X$, $c(S \cup T) = c(c(S) \cup c(T))$) as a means of *weakening* the condition of rationality in a manner which preserves one of the key properties of rational choice, namely that choice over any subset of X "be independent of the way the alternatives were initially divided up for consideration" [**4**, p. 1080]. Although this concept has received widespread attention in the literature,[3] Plott's result that any unique-valued path independent choice function is rational [**4**, Theorem 5] implies that this notion of path independence cannot be used to obtain any *strict* weakening of rationality if we in addition require that the social decision mechanism be sufficiently well specified so as to yield a unique social choice out of any collection of alternatives.

In an attempt to resolve this conflict, Kalai and Megiddo [**2**] (KM) have recently studied the implications of imposing path independence on "randomized choice functions," i.e. choice functions defined over the set X^* of finite subsets of X (where X is now assumed to be a convex subset of some linear space) and where for any $S \in X^*$, $c(S)$ is a unique (possibly degenerate) probability distribution over the elements of S, hence $c(S) \in CH(S) =$ the convex hull of S. KM have shown that if a randomized choice function exhibits disjoint path independence, that is if for all $S, T \in X^*$ for which $S \cap T = \emptyset$, $c(S \cup T) = c(c(S) \cup c(T))$, then (i) for any $S \in X^*, c(S)$ is a lottery over at most two elements of S, and (ii) if there exist at least three noncolinear alternatives in X, then $c(S)$ cannot be continuous (i.e. $c(\{x, y\})$ will not be continuous in x and y).

Such restrictive results might at first be surprising in light of the apparent reasonableness of imposing path independence on randomized choice functions. It is the purpose of this note, however, (a) to show that this concept is even more restrictive than KM suggest, in that if X contains three noncolinear points, then no preference relation on X which generates a unique-valued path independent randomized choice function can be continuous over any open set in X (thus, far from weakening the condition of rationality, it for all practical purposes rules it out),[4] and (b) to give a direct argument that, quite apart from its strong implications, path independence is an immediately unappealing property to impose on randomized choice.

To see the former point, let X be the convex hull of the noncolinear points x, y, and z (so X is the set of probability measures on $\{x, y, z\}$) and assume $c(\cdot)$ is generated by a preference relation $\geqq$ which is continuous on some open set U in X. Since $c(\cdot)$ must be unique-valued (as KM specify), the indifference curves of $\geqq$ over U must be strictly

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[2] A (transitive) preference relation $\geqq$ on X will be said to generate or "rationalize" a choice function $c(\cdot)$ if for all $S \subseteq X$, $c(S) = \{z^* \in S \mid z^* \geqq z \text{ for all } z \in S\}$.

[3] See, for example, Parks [**3**], Sen [**5**], and the references cited there.

[4] A (transitive) preference relation $\geqq$ on X is said to generate or rationalize a *randomized* choice function $c(\cdot)$ if $c(S) = \{z^* \in CH(S) \mid z^* \geqq z \text{ for all } z \in CH(S)\}$ for $S \in X^*$. Note that since continuous preference relations can generate discontinuous choice functions (randomized or otherwise), result (a) is quite distinct from KM's result (ii) above.

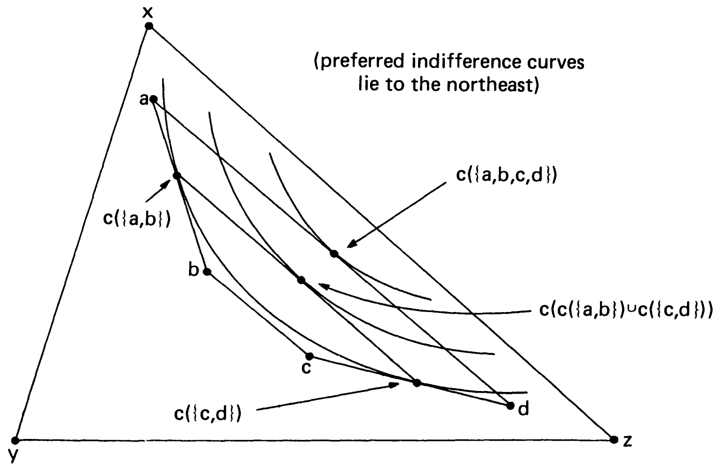

FIGURE 1

convex and not thick. However, from Figure 1 it is clear that in this case we may pick points a, b, c, and d such that $c(c(\{a,b\}) \cup c(\{c,d\})) \neq c(\{a,b,c,d\})$, violating path independence. Hence, a randomized choice function cannot simultaneously be unique-valued, path independent, and rationalized by a preference relation which is continuous over any open set, although it can satisfy any two of these three properties.[5]

Figure 1 also illustrates why path independence is actually an unappealing property to impose on randomized choice functions. Recall that the normative appeal of path independence in the sense of Plott is that a given set of alternatives may be split up into an *exhaustive* partition of smaller sets and choice made over each of these sets, with the results collected and chosen over, and leading to the same result as if choice had been made directly over the original set. However, it is clear from Figure 1 that partitioning the set $\{a,b,c,d\}$ (i.e. the *vertices* of the original attainable set $CH(\{a,b,d,c\})$) into $\{a,b\}$ and $\{c,d\}$, choosing $c(\{a,b\})$ from $CH(\{a,b\})$, $c(\{c,d\})$ from $CH(\{c,d\})$, and then $c(c(\{a,b\}) \cup c(\{c,d\}))$ from $CH(c(\{a,b\}) \cup c(\{c,d\}))$ leaves practically all of the original attainable set $CH(\{a,b,c,d\})$ (i.e. all except for the set $CH(\{a,b\}) \cup CH(\{c,d\}) \cup CH(c(\{a,b\}) \cup c(\{c,d\})))$ completely unconsidered.

The above discussion suggests that a more appealing normative condition might be to replace the domain of $c(\cdot)$ by the set X^{**} of all convex subsets of X and require that, for all $S, T \in X^{**}$ such that $S \cup T$ is also in X^{**}, $c(S \cup T) = c(c(S) \cup c(T))$.[6] However, such a condition seems too weak to have any interesting implications, since relatively few S, T pairs will satisfy $S \cup T \in X^{**}$, nor does it lead to any real economy of choice, since

[5]Note that if $\geqq$ is continuous and strictly quasiconvex it will generate a randomized choice function which satisfies path independence and properties (i) and (ii) above, but which is not unique-valued (thus it satisfies property (ii) in the sense that $c(\cdot)$ will not be a continuous correspondence). Also note that there will exist a lexicographic preference relation on X which, though nowhere continuous in the interior of X, generates a unique-valued randomized choice function satisfying path independence (as well as properties (i) and (ii)).

[6]Note that this formulation differs from the one in Plott [4] in that the domain X^{**} is neither finite nor equal to the entire power set $P(X)$ of X. Equivalently, we may extend $c(\cdot)$ to $P(X)$ by defining $c(S) = c(CH(S))$ for $S \subseteq X$, in which case the condition $c(S \cup T) = c(c(S) \cup c(T))$ is assumed to apply only when $CH(S) \cup CH(T) = CH(S \cup T)$.

if $S \cup T$ is in X^{**}, then either S or T must still be either a singleton or uncountably infinite.

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